skema BUSINESS SCHOOL

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AUCTION THEORY
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Outline – Single Object Auctions

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1. INTRODUCTION

1. Introduction

- Auctions have a long history:
  - Herodotus reports that auctions where used in Babylon 500 years BC for a variety of objects.
  - In 193 AC, the Praetorian Guard, after killing the Emperor Pertinax, sold the entire Roman Empire by auction. The winner was beheaded two months later, an extreme case of “winner’s curse”!
  - Today, auctions are heavily used:
    - Arts and antiques
    - Commodities (tobacco, fish, flowers, …)
    - Financial instruments (US Treasury securities)
    - Privatization (transfer of assets from public to private hands)
    - Auctions of rights (communication spectrum)
    - Internet auctions (Ebay and the like)

1.Introduction

- 1.1 Some Common Auction Forms
  - English auction: open ascending price (most prevalent auction)
    - The price rise continuously (or by small increments).
    - Each bidder indicates an interest at the current price in a manner apparent to all (eg, raising a hand).
    - Once a bidder finds the price to be too high, he signals he is no long interested by lowering his hand.
    - The auction ends when only a single bidder is still interested.
    - The bidder wins et pays the auctioneer an amount equal to the price at the second-last bidder dropped out.
  - Dutch auction: open descending price (not commonly used).
  - The sealed-bid first-price auction (FPA):
    - Bidders submit bids in sealed envelopes.
    - The highest bidder wins et pays his bid.
  - The sealed-bid second-price auction:
    - The bidder pays the second-highest bid.

1. Introduction

- 1.2 Valuations
  - The seller is unsure about the values that bidders attach to the object (the maximum amount the bidder is willing to pay).
  - Different possible situations:
    - Each bidder knows the value of the object to himself at the time of bidding (**private value**). This is typically the case when the value of the object is derived from its consumption or use alone.
    - In many situation, the bidder has only an estimated of some sort or **some** privately known **signal** that is correlated with the true value. Values are unknown at the time of the auction and may be affected by information available to other bidders (**interdependent values**). This is typically the case for objects viewed as investments (resale price matters!).

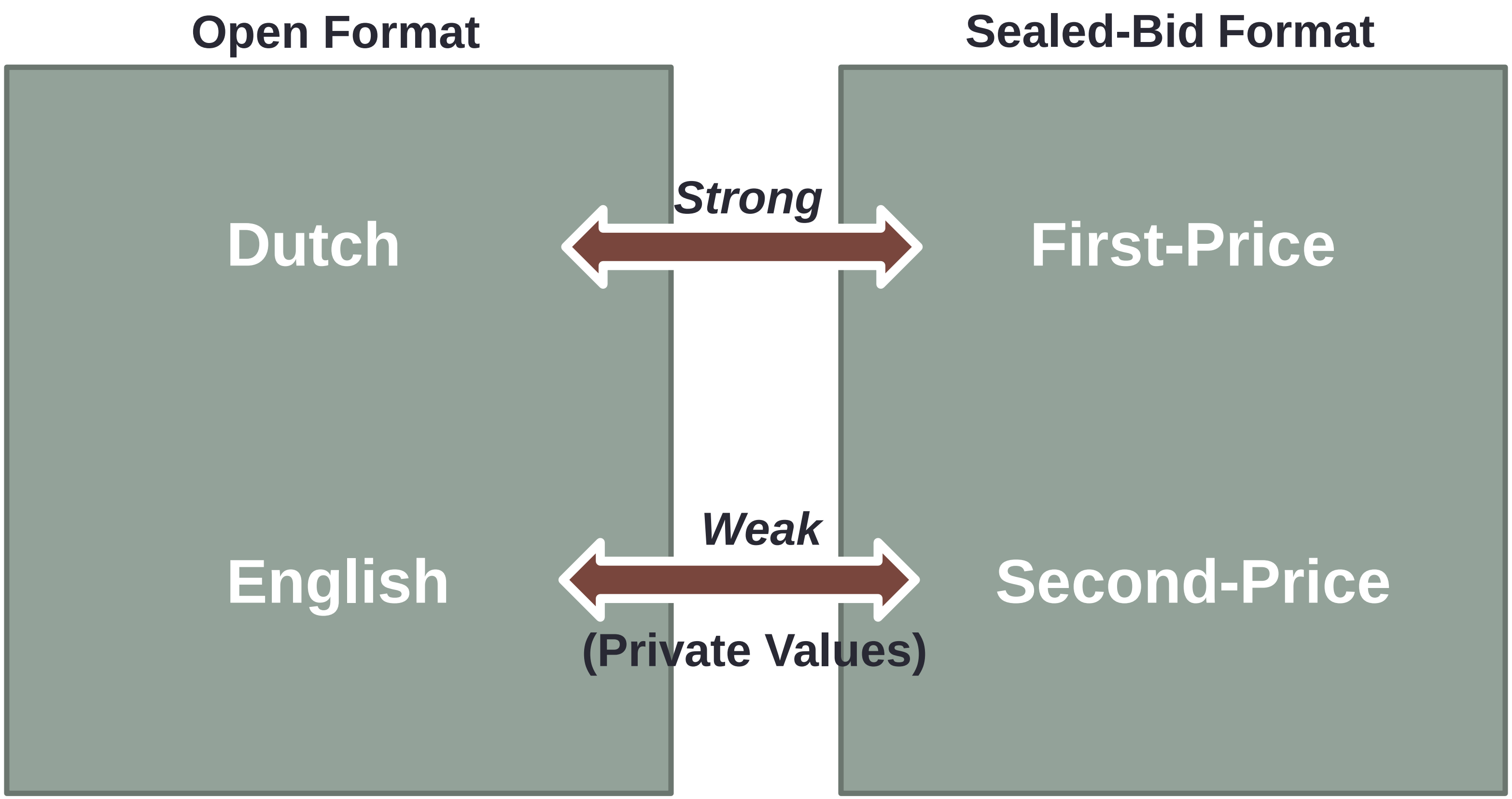
1. Introduction

- If the value of the object is the same for all bidders, we talk about **pure common value**. The classic case is the sale of a track of land with an unknown amount of oil (same market price for all bidders but unknown quantity of oil).

- Note that interdependence:
  - Refers only to the structure of values;
  - Doesn’t refer to any statistical properties of the information held by bidders (independence of signals, correlations, etc)

1. Introduction

- 1.3 Equivalent Auctions



1. Introduction

- **Strong equivalence** (strategic equivalence): same information set to bidders whatever the valuation structure

- **Weak equivalence**:
  - Differences in information sets available to bidders;
  - But the differences in information set areirrelevant under private values;
  - Thus, the optimal strategies are the same **only if values are private**.

1. Introduction

- 1.4 Revenue versus Efficiency
- Main criteria:
  - **Revenue**: expected selling price
  - **Efficiency**: object ends up in the hands of the person who value it the most ex post
  - **Does efficiency matter?** Why can we not rely on the market to reallocate the object efficiently after the auction?

- Other criteria:
  - Simplicity: clear rules
  - Potential bidders collusion

1. Introduction

- 1.5 What is an auction?

  - A wide variety of selling procedure but some common features:
    - Auctions are **universal**:
      - The auction form does not depend on any details specific to the item sold.
      - Auction can therefore be used to sell any good.
    - Auctions are **anonymous**:
      - The identities of the bidders play no role in determining who wins the object and who pays how much.

  - **Mechanisms** differ from auctions in that they are necessarily universal and/or anonymous.

2. PRIVATE VALUE AUCTIONS: A FIRST LOOK

2. Private Value Auctions

- Analysis of the bidding behavior in the four common auction forms (**EN, DUTCH, FPA, SPA**) in an environment with independently and identically distributed (**IID**) private values.

- We introduce the basic methodology in auction theory:
  - Postulate an **informational environment**:
    - **Valuation** structure for the bidders
    - Distribution of **information** available to the bidders
  - Specify the **auction format**

- Each auction format determines a **game of incomplete information**. We search for a **Bayesian-Nash equilibrium** of the game. If there are many, we select one on some basis (dominance, perfection, **symmetry**)

2. Private Value Auctions

- 2.1 The Symmetric Model
  - N potential buyers
  - One object
  - Bidder i assigns a value X_i to the object (the maximum amount the bidder is willing to pay for the object)
    - X_i are **IID** some $[0, \omega)$ according to an increasing CDF F
    - F has a continuous density f with full support
    - We allow that the support is $[0, \omega]$ but then $E[X_i] < \omega$
  - Bidder i knows the realization x_i of X_i and that other bidders' values are IID according to F (**symmetric bidders**)
  - Bidders are **risk neutral** (maximize expected profits).
  - F and N are common knowledge.
  - Bidders are **not subject to any liquidity or budget constraints** (each bidder is willing and able to pay his value x_i).

2. Private Value Auctions

- We examine FPA & SPA:
  - Each auction format determines a game among bidder.
  - A strategy for a bidder is a function $\beta: [0, \omega) \rightarrow R_+$ which determines his bid for any value.
  - We are interested in symmetric equilibrium (all bidders follow the same strategy).

2. Private Value Auctions

- 2.2 Second-Price Auctions
  - Less familiar than FPA but simpler to analysis
  - In the private value setup, equivalent to EN

  - Bidders' payoffs are:

$$\pi_i = \begin{cases} x_i - max_{j \neq i} b_j & \text{if } b_i > max_{j \neq i} b_j \\ 0 & \phantom{\text{if }} b_i < max_{j \neq i} b_j \end{cases}$$

  - In case of tie ($b_i = max_{j \neq i} b_j$), the object goes to each winning bidder with equal probability.

2. Private Value Auctions

- **Proposition 2.1.** *In a second-price sealed-bid auction, it is a weakly dominant strategy to bid according to* $\beta^{II}(x) = x$

- *Proof*.
  - Consider bidder 1 and suppose that $p_1 = max_{j \neq i} b_j$ is the highest competing bid.
  - By bidding x_1, bidder 1 will win if $x_1 > p_1$ and lose if $x_1 < p_1$.
  - Suppose bidder 1 bids $z_1 < x_1$:
    - If $x_1 > z_1 > p_1$: bidder 1 still wins and his profit is still $x_1 - p_1$.
    - If $p_1 > z_1 < x_1$: bidder 1 still loses.
    - If $x_1 > p_1 > z_1$: bidder 1 loses, whereas he had made a positive profit with bid x_1.

    Bidding less than x_1 is (weakly) dominated (a similar argument hold for bidder more than x_1).

2. Private Value Auctions

- Note that proof of Proposition 2.1 relies:
  - Neither on assumption that bidders' values were independently distributed.
  - Nor the assumption that they were identically so.

  Only the private values assumption is important.

- How much each bidder expects to pay in equilibrium?
  - Let $Y_1 \equiv {Y_1}^{(N-1)}$ denotes the highest value among $N-1$ remaining bidders (the highest of $X_2, \ldots X_N$).
  - Let G denote the CDF of Y_1 (with IID assumption, $\forall y, \mathrm{G}(y) = F(y)^{N-1}$.
  - The expected payment is therefore:

$$m^{II}(x) = G(x) \times E(Y_1 | Y_1 < x) \tag{2.1}$$

2. Private Value Auctions

- 2.3 First-Price Auctions
  - In a FPA, the payoffs are

$$\pi_i = \begin{cases} x_i - b_i & \text{if } b_i > max_{j \neq i} b_j \\ 0 & b_i < max_{j \neq i} b_j \end{cases}$$

  - Note that:
    - No bidder would bid an amount equal to his value (this would guarantee a payoff of 0).
    - Each bidder faces a trade-off: an increase in his bid:
      - Increases the probability of winning.
      - Reduces the gains from winning.

2. Private Value Auctions

- Heuristic derivation of symmetric equilibrium strategies:
  - Suppose that bidders $j \neq 1$ follow the symmetric, increasing, and differentiable equilibrium strategy $\beta^I \equiv \beta$.
  - Support bidder 1 receives a signal $X_1 = x$ and bids b.

To derive the optimal b:

- First note that it can never be optimal to bid $b > \beta(\omega)$: bidder 1 would win for sure but could do better by reducing his bid slightly. So, We only consider bids $b \leq \beta(\omega)$.
- Second, a bidder with value 0 would never submit a positive bid (he would make a loss in case of winning). So, $\beta(0) = 0$.

2. Private Value Auctions

- Bidder 1 wins whenever $max_{j \neq i} \beta(X_i) < b$.
- Since β is increasing, $max_{j \neq i} \beta(X_i) = \beta\left(max_{j \neq i} X_i\right) = \beta(Y_1)$.
- Bidder 1 wins therefore whenever $\beta(Y_1) < b$ or $Y_1 < \beta^{-1}(b)$.
- His expected payoff is:

$$G(\beta^{-1}(b)) \times (x - b)$$

- Maximizing with respect to b yields the FOC:

$$\frac{g\left(\beta^{-1}(b)\right)}{\beta'(\beta^{-1}(b))}(x - b) - G(\beta^{-1}(b)) = 0 \qquad (2.2)$$

where g is the PDF of Y_1.

2. Private Value Auctions

- At a symmetric equilibrium, $b = \beta(x)$. Equation (2.2) yields:

$$G(x)\beta'(x) + g(x)\beta(x) = xg(x) \tag{2.3}$$

or

$$\frac{d}{dx}\big(G(x)\beta(x)\big) = xg(x)$$

- Since $\beta(0) = 0$:

$$\beta(x) = \frac{1}{G(x)}\int_0^x yg(y)dy = E[Y_1|Y_1 < x]$$

Note that the derivation is heuristic because (2.3) is a only a necessary condition. Proposition 2.2 establishes it formally.

2. Private Value Auctions

- **Proposition 2.2.** *Symmetric equilibrium strategies in a first-price auction are given by*

$$\beta^I(x) = E[Y_1 | Y_1 < x] \quad (2.4)$$

where Y_1 is the highest of $N - 1$ independently drawn values.

- *Proof*.
  - Suppose that all but bidder 1 follow the strategy $\beta^I(x) \equiv \beta$.
  - It is then optimal for bidder 1 to follow also $\beta^I(x)$:
    - First, note that β is an increasing and continuous function. This, in equilibrium, the bidder with the highest value submits the highest bid and wins the auction.
    - Second, it is never optimal for bidder 1 to bid $b > \beta(\omega)$.

2. Private Value Auctions

- Denote $z = \beta^{-1}(b)$ the value for which b is the equilibrium bid ($b = \beta(z)$). The bidder 1's expected payoff when his value is x is:

$$\pi(b, x) = G(z)[x - \beta(z)]$$
$$\pi(b, x) = G(z)x - G(z)E[Y_1|Y_1 < x]$$
$$\pi(b, x) = G(z)x - \int_0^z yg(y)dy$$

By integration by parts, we get:

$$\pi(b, x) = G(z)x - G(z)z + \int_0^z G(y)dy$$
$$\pi(b, x) = G(z)(x - z) + \int_0^z G(y)dy$$

We obtain there the deviation from bidding according to x:

$$\pi(\beta(x), x) - \pi(\beta(z), x) = G(z)(z - x) + \int_z^z G(y)dy \geq 0$$

(see surfaces computation to see why when $z \geq x$ and $z \leq x$).

2. Private Value Auctions

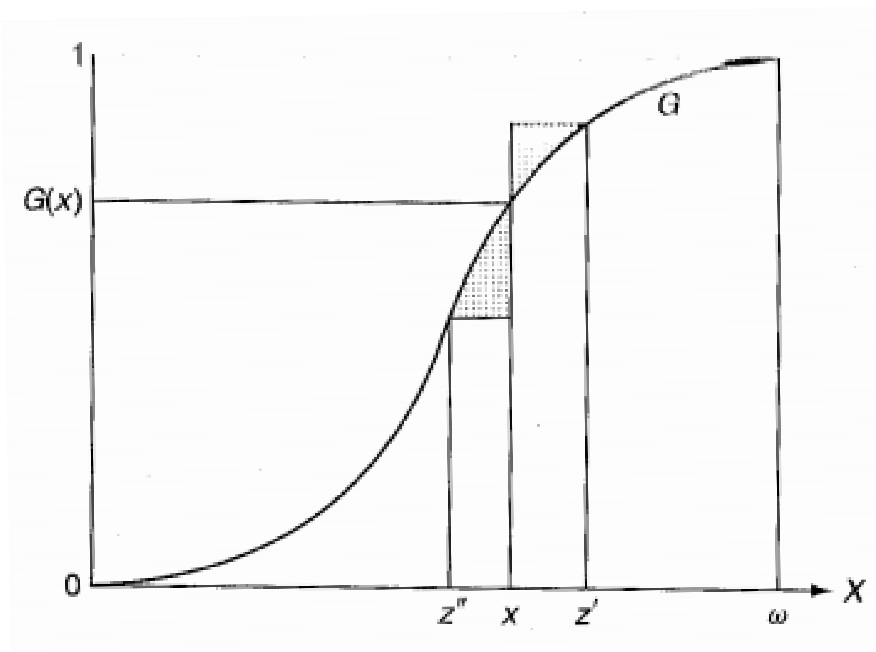


FIGURE 2.1 Losses from over- and underbidding in a first-price auction.

- If all bidders follow β, a bidder with value x cannot benefit from bidding anything other then $\beta(x)$.

2. Private Value Auctions

- Using integration by parts, the equilibrium bidding strategy can rewritten:

$$\beta(x) = \frac{1}{G(x)} \int_0^x y g(y) dy$$

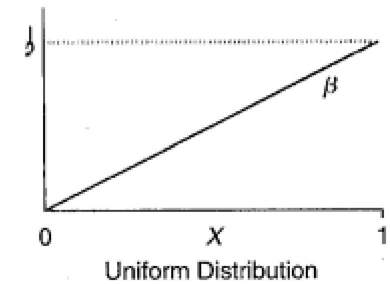
$$\beta(x) = \frac{1}{G(x)} \left([yG(y)]_0^x - \int_0^x G(y) dy \right)$$

$$\beta(x) = x - \int_0^x \frac{G(y)}{G(x)} dy \leq x$$

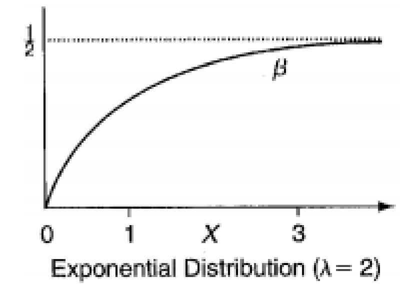
- Note also that with IID values, $\frac{G(y)}{G(x)} = \left[\frac{F(y)}{F(x)}\right]^{N-1}$: the degree of bid shading depends on the number of competing bidders (as N increases, $\left[\frac{F(y)}{F(x)}\right]^{N-1}$ approaches 0 and $\beta(x)$ approaches x.

2. Private Value Auctions

- **Example 2.1.** Values are uniform on [0,1]
  - $F(x) = x$
  - $G(x) = x^{N-1}$
  - $\beta^I(x) = \frac{N-1}{N} x$



- **Example 2.2**. Values are exponentially distributed on $[0, \infty)$ and $N = 2$
  - $F(x) = 1 - \exp(-\mu x)$ for $\mu > 0$
  - $G(x) = F(x)$
  - $\beta^I(x) = x - \int_0^X \frac{F(y)}{F(x)} dy = \frac{1}{\mu} - \frac{x \exp(-\mu x)}{1-\exp(-\mu x)}$
  - Note that for the exponential distribution, $E(x) = \frac{1}{\mu}$
  - Note also that, by conditional expectation, $\beta^I(x) = E[Y_1 | Y_1 < x] \leq \mathrm{E}[Y_1]$. If $\mu = 2$ and $N = 2$, $\mathrm{E}[Y_1] = {}^1/_2$.

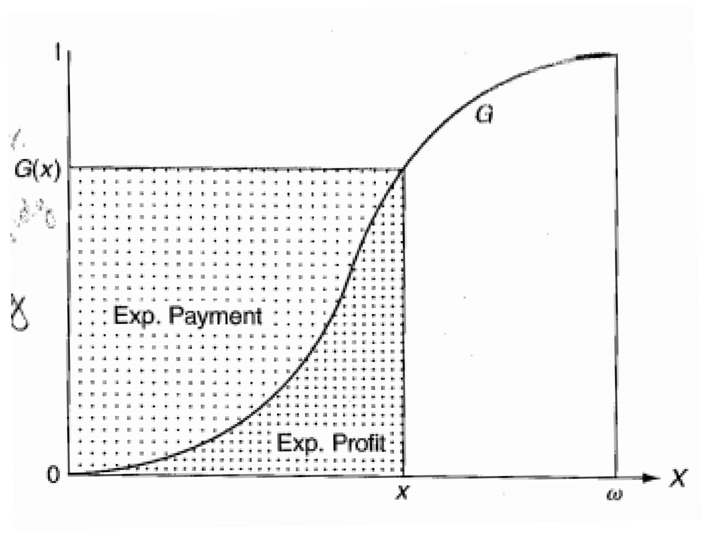


2. Private Value Auctions

- 2.4 Revenue Comparison
  - We will now compare selling prices in FPA and SPA.
  - FPA: probability to win x bid amount

$$m^I(x) = G(x) \times [Y_1 | Y_1 < x] \tag{2.5}$$

  - This is the same as in the SPA (see (2.1)).
  - Figure 2.3 depicts the expected payment and the expect payoff of a bidder



2. Private Value Auctions

- Expected revenues in the 2 auctions are the same in FPA and SPA:
  - In both auctions, *ex ante* expected payment is:

$$E[m^A(x)] = \int_0^\omega m^A(x) f(x) dx$$

$$= \int_0^\omega \left(\int_0^x y g(y) dy \right) f(x) dx$$

1. By interchanging the order of integration, we get:

$$E[m^A(x)] = \int_0^\omega \left(\int_y^\omega f(x) dx \right) y g(y) dy$$

$$= \int_0^\omega y(1 - F(y)) g(y) dy \qquad (2.6)$$

- The expected revenue of the seller is N times the *ex ante* expected payment of one bidder:

$$E[R^A(x)] = N \int_0^\omega y(1 - F(y)) g(y) dy$$

2. Private Value Auctions

- But because the density of $Y_2^{(N)}$, the second highest of N values, is $f_2^{(N)}(y) = N(1 - F(y))f_1^{(N-1)}(y)$ and since $f_1^{(N-1)}(y) = \mathrm{g}(y)$:

$$E[R^A(x)] = \int_0^{\omega} y f_2^{(N)}(y)dy = E[Y_2^{(N)}] \qquad (2.7)$$

1. So, in either case, the expected revenue is just the expectation of the second-highest value.

- **Proposition 2.3.** *With IID private values, the expected revenue in a FPA is the same as the expected revenue in a SPA.*

2. Private Value Auctions

- Note that the fact that the expected selling prices are equal is really striking because in specific realizations of the values, the price at which the object is sold may be greater in one auction of the other.

- Eg.: IID uniform values, $N = 2$
  - FPA: $\beta^{I}(x) = \frac{1}{2}x$
  - SPA: $\beta^{II}(x) = x$
  - So, if $\frac{1}{2}x_1 > x_2$: FPA > SPA
  - But if $\frac{1}{2}x_1 < x_2 < x_1$: SPA > FPA

2. Private Value Auctions

- Note also the revenues in a SPA are more variable than in a FPA:
  - In SPA, the price can range from 0 to ω
  - In FPA, the price can range from 0 to $E[Y_1]$ (because x range from 0 to ω).

- If L^I is the distribution of equilibrium price in a FPA and L^{II} of the SPA, L^{II} is a mean preserving spread of L^I. Therefore, any risk-averse seller (assuming bidder are risk-neutral) will prefer FPA to SPA.
- *Proof*.
  - The revenue in the SPA is the random variable $R^{II} = {Y_2}^{(N)}$
  - The revenue in the FPA is the random variable $R^I = \beta({Y_1}^{(N)})$ (where β is the symmetric equilibrium from Proposition 2.2)
  - We can therefore write:

$$E[R^{II}|R^I = p] = E\left[{Y_2}^{(N)}\middle|{Y_1}^{(N)} = \beta^{-1}(p)\right]$$

  - But ∀ y:

$$E\left[{Y_2}^{(N)}\middle|{Y_1}^{(N)} = \mathrm{y}\right] = \mathrm{E}[{Y_1}^{(N-1)}|{Y_1}^{(N-1)} < y] \quad (2.8)$$

(because the only information that the event that the highest of N values equal y provides about the second highest of N values is that the highest of $N - 1$ values is less than y.

2. Private Value Auctions

- Using (2.8) and (2.4), we can write:

$$E[R^{II}|R^{I} = p] = E\left[Y_1^{(N-1)} \middle| Y_1^{(N-1)} < \beta^{-1}(p)\right]$$

$$= \beta(\beta^{-1}(p)) = p$$

- Because $E[R^{II}|R^{I} = p] = p$, there exists a random variable Z such that the distribution of R^{II} is the same of that of $R^{I} + Z$ and $E[Z|R^{I} = p] = 0$. R^{II} is a mean preserving spread of R^{I}.

2. Private Value Auctions

- 2.5. Reserve Prices
  - The seller reserves the right to not sell if the price is lower than $r > 0$.
  - SPA:
    - No bidder with value $x < r$ can make a positive profit.
    - But it is still weakly dominant strategy to bid one's value.
    - The expected payment of a bidder with value $x \geq r$ is:

$$m^{II}(x,r) = rG(r) + \int_r^x y g(y) dy \qquad (2.9)$$

(the winner pays r when the second highest value is below r)

2. Private Value Auctions

- FPA
  - No bidder with value $x < r$ can make a positive profit.
  - If β^I is a symmetric equilibrium of FPA with reserve price r, it must be that $\beta^I(r) = r$: a bidder with value r wins only if all other bidders have values less than r (in that case, he wins with a bid of r).
  - In an analogous development of Proposition 2.2, we obtain the symmetric equilibrium bidding strategy for bidder with value $x \geq r$:

$$\beta^I(x) = r\frac{G(r)}{G(x)} + \frac{1}{G(x)}\int_r^x yg(y)dy$$
$$= E[\max(Y_1, r)\,|Y_1 < x]$$

  - The expected payment is:

$$m^I(x,r) = G(x) \times \beta^I(x) = rG(r) + \int_r^x yg(y)dy \qquad (2.10)$$

2. Private Value Auctions

- Revenue Effects of Reserve Prices
  - Let A denote either FPA or SPA.
  - In both, the expected payment of a bidder with value r is $r\,G(r)$.
  - As for (2.6), the *ex ante* expected payment is:

$$E[m^A(X,r)] = \int_r^\infty m^A(x,r)f(x)dx$$
$$= r\big(1 - F(r)\big)G(r) + \int_r^\infty y\big(1 - F(y)\big)g(y)dy$$

  - To derive the optimal reserve price, suppose that the seller attaches a value $x_0 \in [0,\infty)$ to the good:
    - The seller will not set $r < x_0$.
    - The expected payoff of the seller from setting $r \geq x_0$ is:

$$\pi_0 = N \times E[m^A(X,r)] + F(r)^N x_0$$

2. Private Value Auctions

- Differentiating with respect to r, we obtain the FOC:

$$\frac{\partial \pi_0}{\partial r} = N[1 - F(r) - rf(r)]G(r) + NG(r)f(r)x_0$$

- Now recalling that the hazard rate function associate with CDF F is $lambda(x) = {}^{f(x)}/_{1-F(x)}$, we can write:

$$\frac{\partial \pi_0}{\partial r} = N[1 - (r - x_0)lambda(r)]\big(1 - F(r)\big)G(r) \qquad (2.11)$$

- Note that:
  - If $x_0 > 0$, the derivative at $r = x_0$ is positive: the seller should set a reserve price $r > x_0$.
  - If $x_0 = 0$, the derivative at $r = 0$ is 0 (because $G(0) = 0$). As long as $lambda(r)$ is bounded, the expected payment attains a local minimum at 0. A small r leads to an increase in revenue.

2. Private Value Auctions

- Conclusion: **a revenue maximizing seller should always set a reserve price that exceeds his or her value**. This is refereed to as the **exclusion principle**: it is optimal for the seller to exclude some bidders from the auction, even if their values exceed x_0.

- Note also that, the FOC defined by Equation (2.11) implies that $[1-(r-x_0)lambda(r)]$ is equal to 0 or that the optimal reserve price r^* satisfies:

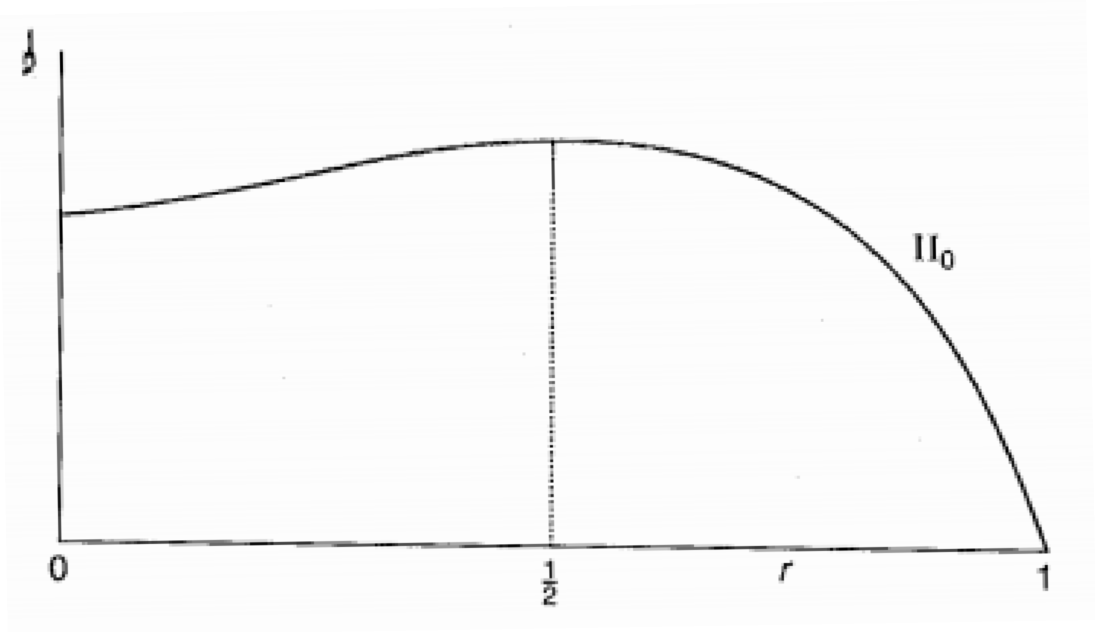
$$(r^* - x_0)lambda(r^*) = 1$$

$$r^* - \frac{1}{lambda(r^*)} = x_0 \qquad (2.12)$$

- If $lambda(.)$ is increasing, this condition is also sufficient and it is remarkable that the optimal reserve price does not depend on the number of bidders. The intuition is that reserve price comes into play only when there is a single bidder with a value that exceeds the reserve price.

2. Private Value Auctions

- Eg.: Expected revenue as a function of the reserve price when F is uniform on $[0,1]$, $N = 2$ and $x_0 = 0$.



2. Private Value Auctions

- Entry Fees
  - Entry fee: a fixed and nonrefundable amount that bidders must pay the seller prior to the auction in order to be able submit bids.
  - An alternative instrument that the seller can use to exclude buyers with low values:
    - A reserve of r excludes bidders with values $x < r$.
    - The same set of bidders can be excluded by asking an entry fee $e = G(r) \times r$: after paying e, the expected payoff of a bidder with value r is exactly 0.

- Efficiency versus Revenue
  - A reserve price (and an entry fee) raises the revenue to the seller but may lead to inefficient allocation: if $x_0 = 0$ and $r > 0$, the object may remain unsold despite some bidders may have positive valuation.

- Commitment issues
  - The analysis of reserve assumes that the seller can credibly commit to not sell the object if it cannot be sold at or above the reserve price. If the commitment is not credible, this will be anticipated by bidders who will adjust their bidding behavior.

2. Private Value Auctions

- Problems
  - 2.1. Power distribution
  - 2.2. Pareto distribution
  - 2.3. Stochastic dominance
  - 2.4. Mixed auction
  - 2.5. Resale

3. THE REVENUE EQUIVALENCE PRINCIPLE

The Revenue Equivalence Principle

- In Chapter 2, we saw that the expected selling price in the FPA and the SPA is the same in a symmetric private value setup.
- We explore here this remarkable result.
- Note that the expected selling price equality doesn’t mean that *ex-post*, the selling prices will be the same.

The Revenue Equivalence Principle

- 3.1. Main Result
  - Standard auction:
    - The rule of the auctions dictate that the person who bids the highest amount is awarded the object.
    - Examples: FPA, SPA.
    - Non standard auctions: lottery (the probability to win is a function of the ratio of the bid to the total amount bid by all participants)
  - Denote:
    - A a standard auction.
    - β^A the symmetric equilibrium of the auction.
    - $m^A(x)$ the equilibrium expected payment by a bidder with value x.

The Revenue Equivalence Principle

- **Proposition 3.1.** *Suppose that values are idependently and identically distributed and all bidders are risk neutral. Then any symmetric and increasing equilibrium of any standard auction, such that the expected payment of a bidder with value zero is zero, yields the same expected revenue to the seller.*

- *Proof*.
  - Consider a standard auction A.
  - Fix a symmetric equilibrium β.
  - Let $m^A(x)$ be the equilibrium expected payment in auction A.
  - Suppose β is such that $m^A(0) = 0$.

The Revenue Equivalence Principle

- Consider bidder 1 with true value x and assume all other bidders follow the equilibrium bidding strategy.
- What happens if bidder 1 bids according to $\beta(z)$ instead of $\beta(x)$, this is to say deviates from the equilibrium:

  - Bidder 1 expected payoff:

$$\pi^A(z, x) = G(z)x - m^A(z)$$

where $G(z) \equiv F(z)^{N-1}$ is the probability that bidder 1 will win the auction. Note $m^A(z)$ depends on other players' strategy β and z is independent of x.

  - First order condition is

$$\frac{\partial}{\partial z}\pi^A(z, x) = g(z)x - \frac{\partial}{\partial z}m^A(z) = 0$$

The Revenue Equivalence Principle

- At equilibrium, it is optimal to report $z = x$ (Revelation Principle, Myerson (1981)):

$$\frac{\partial}{\partial y} m^A(y) = g(y)y \tag{3.1}$$

- By the Fundamental Theorem of Integration:

$$m^A(x) = m^A(0) + \int_0^x yg(y)dy = \int_0^x yg(y)dy$$

$$m^A(x) = G(x) \times E[Y_1 | Y_1 < x] \tag{3.2}$$

- The right-hand side of Equation (3.2) doesn't depend on a particular auction form.

Cqfd.

The Revenue Equivalence Principle

- **Example 3.1.** *Values are uniformly distributed on* [0,1]
  - In this case, $F(x) = x$ and $G(x) = x^{N-1}$
  - Therefore:

$$m^A(x) = \frac{N-1}{N} x^N$$

$$E[m^A(x)] = \int_0^1 m^A(x) f(x) dx = \frac{N-1}{N(N+1)}$$

$$E[R^A] = N \times E[m^A(x)] = \frac{N-1}{N+1}$$

The Revenue Equivalence Principle

- 3.2. Some Applications
  - 3.2.1. Unusual Auctions

EQUILIBRIUM OF ALL-PAY AUCTIONS

- Rules:
  - Each bidder submit a bid.
  - The highest bidder wins.
  - All bidders pay their bid!

- Application: model of lobbying activity. The money spend on lobbying is a sunk cost.

- We are interested in symmetric equilibrium strategies.

The Revenue Equivalence Principle

- Suppose there is a symmetric increasing equilibrium of the all-pay auction such that $m^A(0) = 0$. So, conditions of Proposition 3.1 are satisfied.

- In the all-pay (AP) auction, the expected payment of a bidder with value x is his bid. So, it must be that:

$$\beta^{AP}(x) = m^A(x) = \int_0^x yg(y)dy$$

by Proposition 3.1.

- To verify that it is an equilibrium, suppose that all bidders except one are following the strategy $\beta \equiv \beta^{AP}$.

The Revenue Equivalence Principle

- If the bidder bids according to $\beta(z)$, is expected payoff is:

$$G(z)x - \beta(z) = G(z)x - \int_0^z yg(y)dy$$

- Integrating the second term by pars, we obtain:

$$G(z)x - [yG(y)]^z{}_0 - \int_0^z G(y)dy$$

- This is the same payoff as in the FPA. So, as for Proposition 2.2, it is maximized at $z = x$ and β^{AP} is a symmetric equilibrium of AP.

The Revenue Equivalence Principle

EQUILIBRIUM OF THIRD-PRICE AUCTIONS

- Rules:
  - Each bidder submit a bid.
  - The highest bidder wins.
  - The winner pays a price equal to the third-highest bid (N must be equal at least to 3).

- No known instance of such a mechanism.

- Suppose there is a symmetric, increasing equilibrium of the third-price auction β^{III} such that $m^{A}(0) = 0$. So, Proposition 3.1 applies.

- By Proposition 3.1:

$$m^{III}(x) = \int_0^x y g(y) dy \tag{3.3}$$

The Revenue Equivalence Principle

- But suppose Bidder 1 wins when his value is x:
  - Winning implies that $Y_1 < x$.
  - The price he pays is $\beta^{III}(Y_2)$ where Y_2 is the second highest of the $N-1$ other values.
  - The density of Y_2 conditional on the event that $Y_1 < x$ is:

$$f_2^{(N-1)}(y|Y_1 < x) = \frac{1}{F_1^{(N-1)}(x)}(N-1)\big(F(x) - F(y)\big)f_1^{(N-2)}(y)$$

where:

- $F_1^{(N-1)}(x)$ is the probability of the condition.
- $(N-1)\big(F(x) - F(y)\big)$ is the probability that Y_1 exceeds $Y_2 = y$ but is less than x (see relation C.5)
- $f_1^{(N-2)}(y)$ is the density of the highest of $N-2$ values.

The Revenue Equivalence Principle

- The expected payment is:

$$m^{III}(x) = F_1^{(N-1)}(x)E[\beta^{III}(Y_2)|Y_1 < x]$$

$$m^{III}(x) = \int_0^x \beta^{III}(y)(N-1)\big(F(x)-F(y)\big)f_1^{(N-2)}(y)dy \quad (3.4)$$

- Equating (3.3) and (3.4), we obtain:

$$\int_0^x \beta^{III}(y)(N-1)\big(F(x)-F(y)\big)f_1^{(N-2)}(y)dy = \int_0^x yg(y)dy$$

- Differentiating with respect to x, we obtain:

$$(N-1)f(x)\int_0^x \beta^{III}(y)f_1^{(N-2)}(y)dy = xg(x)$$

The Revenue Equivalence Principle

- But $g(x) = (N-1)F(x)^{N-2}f(x)$ because $G(x) = F(x)^{N-1}$. Note also that $F(x)^{N-2} = {F_1}^{(N-2)}(x)$. We obtain therefore:

$$\int_0^x \beta^{III}(y){f_1}^{(N-2)}(y)dy = x{F_1}^{(N-2)}(x)$$

- Differentiating once more with respect to x, we get:

$$\beta^{III}(x){f_1}^{(N-2)}(x) = x{f_1}^{(N-2)}(x) + {F_1}^{(N-2)}(x)$$

- This leads to the equilibrium bidding strategy:

$$\beta^{III}(x) = x + \frac{{F_1}^{(N-2)}(x)}{{f_1}^{(N-2)}(x)} = x + \frac{F^{N-2}(x)}{(N-2)F^{N-3}(x)f(x)} = x + \frac{F(x)}{(N-2)f(x)}$$

The Revenue Equivalence Principle

- This derivation is only valid if $\beta^{III}(x)$ is increasing. A sufficient condition is that $\frac{F(x)}{f(x)}$ is increasing or that the reverse hazard rate $\sigma(x)$ is decreasing. This is equivalent to requiring that $\ln F(x)$ is a concave function (or $F(x)$ is log-concave).

- Note also that the equilibrium $\beta^{III}(x)$ leads to bid OVER is own value! This is due to the fact that is not dominated for a bidder to bid over his own value:
  - Suppose that all bidders except 1 follow $\beta^{III}(x)$
  - Suppose bidder 1 with value x bids $b > x$
  - If $\beta(Y_2) < x < \beta(Y_1) < b$, bidding b is better then bidding x.
  - If $\mathrm{x} < \beta(Y_2) < \beta(Y_1) < b$, bidding b leads to a loss.
  - When $b - x = \varepsilon$ is small, in the first case, ε is of order 2, while in the second case, it is of order 3. It is therefore optimal to bid over his own value.

The Revenue Equivalence Principle

- 3.2.2. Uncertain Number of Bidders
  - Let $\boldsymbol{N} = \{1,2,\ldots N\}$ denote the set of *potential* bidders and $\boldsymbol{A} \subseteq \boldsymbol{N}$ be the set of actual bidders (bidders that participate to the auction).
  - Consider a bidder $i \in \boldsymbol{A}$ and let p_n be the probability that any participating bidder i assigns to the event that he is facing n other bidders.
  - Assume that actual bidders holds the *same* beliefs about how many other bidders he faces (symmetric beliefs). The probability p_n doesn't depend on the identity of the bidder.
  - Note also that the set of actual bidders $\boldsymbol{A}$ doesn't depend on realized values.

As long of the assumption about symmetric beliefs is maintained, conclusion of Proposition 3.1 remains value.

The Revenue Equivalence Principle

- Proof
  - Consider a standard auction and a symmetric and increasing equilibrium β.
  - Since bidders are unsure about the number of bidders n, β doesn't depend on n.
  - Consider the expected payoff of a bidder with value x who bids according to z:
    - The probability that the bidder meet n rivals is p_n.
    - In that case, he wins if ${Y_1}^{(n)}$ (the highest of n values) is less than z. This happens with probability $G^{(n)}(z) = F^n(z)$.
    - The overall probability to win is therefore:

$$G(z) = \sum_{n=0}^{N-1} p_n G^{(n)}(z)$$

The Revenue Equivalence Principle

- His expected payoff is as usual:

$$\pi(x, z) = G(z)x - m(z)$$

- The remainder of the proof is parallel to the proof of Proposition 3.1. The revenue equivalence principle holds even if the number of bidders is uncertain (under our assumptions).

- Example: SPA versus FPA
  - The object is sold in a second-price auction. It remains a dominant strategy to bid his own value. The expected payment is then:

$$m^{II}(x) = \sum_{n=0}^{N-1} p_n G^{(n)}(x) E\left[{Y_1}^{(n)} | {Y_1}^{(n)} < x \right]$$

The Revenue Equivalence Principle

- If the object is sold using a first-price auction, the expected payment of a bidder with value x is

$$m^I(x) = G(z)\beta(x)$$

- By the revenue equivalence principle, $m^I(x) = m^{II}(x)$. So:

$$\beta(x) = \sum_{n=0}^{N-1} \frac{p_n G^{(n)}(x)}{G(x)} E\left[{Y_1}^{(n)} | {Y_1}^{(n)} < x\right] = \sum_{n=0}^{N-1} \frac{p_n G^{(n)}(x)}{G(x)} \beta^{(n)}(x)$$

- $\beta^{(n)}(x)$ is a bidding strategy of first-price auction where there is exactly $n+1$ bidders. So, the equilibrium bidding strategy of an auction with unsure number of bidders is a weighted average of the bidding strategies when the number of bidders is known.

4. QUALIFICATIONS AND EXTENSIONS

Qualifications and Extensions

- Key assumptions of Revenue Equivalence Principle:
  - Independence
  - Risk neutrality
  - No budget constraints
  - Symmetry

- What happens if assumptions are relaxed?
  - Risk averse bidders
  - Budget constraints
  - Heterogeneity among bidders (asymmetry)

- We study here the consequence of relaxing assumptions one by one

Qualifications and Extensions

- **4.1 Risk-Averse Bidders**
  - If bidders are risk averse (but keeping all others assumptions: independence of values, symmetry, no budget constraints), the revenue equivalence principle no longer holds

  - Risk-neutrality implies that bidder's payoff is additively separable:
    - Expected payoff = expected gain – expected payment
    - Payoff is linear in payments
    - Expected payoff is said to be quasi-linear

  - Assume bidders has a von-Neumann-Morgenstern utility function $u: R_+ \to R$ such that $u(0) = 0$, $u' > 0$ and $u'' < 0$ (concave utility function)

Qualifications and Extensions

- **Proposition 4.1.** *Suppose that bidders are risk-averse with the same utility function. With symmetric, independent private values, the expected revenue in a first-price auction is greater that in a second-price auction.*

- *Proof.*
  - Risk-aversion makes not difference in a SPA: it still a weakly dominant strategy to bid his value. So, expected price is the same as it would be if bidders were risk-neutral.

  - To analyze the FPA, assume that the equilibrium strategy is an increasing and differentiable function: $\gamma: [0, \omega] \to R_+$, satisfying $\gamma(0) = 0$.
  - Assume also that all bidders follow γ except bidder 1. Then, bidder 1 will never bid more $\gamma(\omega)$.
  - Given a value x, each bidder has to choose $z \in [0, \omega]$ (and the associated bid amount $\gamma(z)$) to maximize his expected payoff

Qualifications and Extensions

$$max_z \, G(z)u(x-\gamma(z)) \tag{4.1}$$

where $G(z) = F^{N-1}(z)$ is the probability to win.

- The corresponding FOC is:

$$g(z) \times u\big(x-\gamma(z)\big) - G(z) \times \gamma'(z) \times u'\big(x-\gamma(z)\big) = 0$$

- By the revelation principle, it must be optimal to choose $z = x$. We get:

$$\frac{g(x)u(x-\gamma(x))}{\gamma'(x)} = G(x)u'\big(x-\gamma(x)\big)$$

$$\gamma'(x) = \frac{u\big(x-\gamma(x)\big)}{u'\big(x-\gamma(x)\big)} \times \frac{g(x)}{G(x)} \tag{4.2}$$

Qualifications and Extensions

- With risk neutrality, $u(x) = x, u'(x) = 1$ and we get:

$$\beta'(x) = (x - \beta(x)) \times \frac{g(x)}{G(x)} \tag{4.3}$$

- With risk aversion, u is strictly concave and $u(0) = 0$. Therefore, for all $y > 0$, $\frac{u(y)}{u'(y)} > y$:
  - Take the derivative on both sides:

$$\frac{u'(y) \times u'(y) - u(y) \times u''(y)}{(u'(y))^2} > 1$$

$$\frac{-u(y) \times u''(y)}{(u'(y))^2} > 0$$

  - This is the case because $u' > 0$ and $u''(x) < 0$

Qualifications and Extensions

- Using this fact, from (4.2):

$$\gamma'(x) = \frac{u(x-\gamma(x))}{u'(x-\gamma(x))} \times \frac{g(x)}{G(x)} > (x - \gamma(x)) \times \frac{g(x)}{G(x)} \quad (4.4)$$

- If $\beta(x) > \gamma(x)$, $(x - \gamma(x)) \times \frac{g(x)}{G(x)} > (x - \beta(x)) \times \frac{g(x)}{G(x)}$ and, because of (4.4), $\gamma'(x) > \beta'(x)$.

- To summarize, if β is the strategy of the risk neutral bidder and γ the strategy of the risk averse bidder:

$$\beta(x) > \gamma(x) \text{ implies } \beta'(x) < \gamma'(x) \quad (4.5)$$

Qualifications and Extensions

- But:

$$\beta(0) = \gamma(0) \tag{4.6}$$

- (4.5) and (4.6) implies, for all $x > 0$, that $\beta(x) < \gamma(x)$ (if the slope of β is lower than the slope of γ and if $\beta = \gamma$ if $x = 0$, it is impossible for β to be above γ for $x > 0$).

- We conclude that, in a FPA, risk-aversion leads to an increase in bids. Since bids increases, expected revenues increase. As expected revenue in SPA doesn't increase, expected revenue in FPA dominates expected in SPA in case of risk aversion.

- .

Qualifications and Extensions

- Why? With risk aversion, the cost of loosing increases with respect to the cost of loosing in absence of risk aversion (because of the concavity of the utility function). This increases incentives to bid higher. To bid higher is like **buying an insurance** against the risk of loosing

- **Example 4.1.** *Constant relative risk aversion (CRRA) utility*
  - $N = 2$
  - $u(z) = z^{\alpha}$ with $0 < \alpha < 1$
  - Values drawn from F
  - Denote $F_{\alpha} \equiv F^{1/\alpha}$

  - Then:
    - Coefficient of relative risk aversion: $\frac{-z\, u''(z)}{u'(z)} = \frac{-z\alpha\,(\alpha-1)z^{\alpha-2}}{\alpha\, z^{\alpha-1}} = 1-\alpha$
    - $F^{1/\alpha}$ is also a distribution function with same support as F

Qualifications and Extensions

- The symmetric equilibrium in FPA is the solution to (4.2):

$$\gamma'(x) = \frac{u\big(x-\gamma(x)\big)}{u'\big(x-\gamma(x)\big)} \times \frac{g(x)}{G(x)} \qquad (4.2)$$

Note:

- $\frac{u\big(x-\gamma(x)\big)}{u'\big(x-\gamma(x)\big)} = \frac{(x-\gamma(x))^{\alpha}}{\alpha(x-\gamma(x))^{\alpha-1}} = \frac{(x-\gamma(x))}{\alpha}$ because $u(x) = x^{\propto}$ and $u'(x) = \alpha x^{\alpha-1}$
- $\frac{g(x)}{G(x)} = \frac{f(x)}{F(x)}$

- Therefore:

$$\gamma'(x) = \frac{(x-\gamma(x))}{\alpha} \times \frac{f(x)}{F(x)}$$

Qualifications and Extensions

- Or:

$$F(x)\gamma'(x) = \frac{1}{\alpha}xf(x) - \frac{1}{\alpha}\gamma(x)f(x)$$

$$F(x)\gamma'(x) + \frac{1}{\alpha}\gamma(x)f(x) = \frac{1}{\alpha}xf(x)$$

- This is a first order linear differential equation. The solution is:

$$\gamma(x) = \frac{1}{F(x)^{1/\alpha}}\int_0^x y\frac{1}{\alpha}F(y)^{1/\alpha-1}f(y)dy$$

$$\gamma(x) = \frac{1}{F_\alpha(x)}\int_0^x yf_\alpha(y)dy$$

- This is the same form as derived in Proposition 2.2.

Qualifications and Extensions

- So, FPA bidding strategy with 2 CRRA bidders with utility $u(z) = z^{\alpha}$ whose values are drawn from the distribution F is the same as the equilibrium bidding strategy with 2 risk-neutral bidders whose value are drawn from the distribution F_{α}. Since $F_{\alpha} \leq F$ (F_{α} first-order stochastically dominates F), the expected revenue in a FPA with risk-averse bidders is greater than with risk-neutral bidders.

- **Example 4.2.** *Constant absolute risk aversion (CARA) utility functions.*
  - $u(z) = 1 - \exp(-\alpha z)$, with $\alpha > 0$
  - Values are independently distributed according to F
  - G denote the distribution of the highest of $N - 1$ values.

  - Then:
    - Coefficient of absolute risk aversion: $\frac{-u''(z)}{u'(z)} = \frac{-(-\alpha^2 \exp(-\alpha z))}{\alpha \exp(-\alpha z)} = \alpha$

Qualifications and Extensions

- Consider first a SPA:
  - Take the case of a bidder with value x who bids z and wins the auctions.
  - This bidder faces an uncertainty about the price he will pay since that price is determined by the second-highest bid.
  - Suppose the other bidders follow the (weakly) dominant strategy of bidding their values Y_1.
  - Let $\rho(x,z)$ be the risk premium associated with the price gamble (the amount the bidder would forgo to remove the associated uncertainty).:

$$u\big(x-\rho(x,z)\big)=E[u(x-Y_1)|Y_1<z] \qquad (4.7)$$

Note that CARA implies that $\rho(x,z)\equiv\rho(z)$: the utility doesn't depend on the wealth level.

- It is optimal for the bidder 1 to bid his true value (SPA equilibrium). Thus:

$$x\in \arg max_z\ G(z)E[u(x-Y_1)|Y_1<z]$$

Qualifications and Extensions

- Using (4.7), we can write:

  $$x \in \arg max_z \; G(z)u\big(x - \rho(z)\big)$$

- This is the same maximization problem as in a FPA if all bidders follow the strategy $\gamma = \rho$ (see 4.1). Therefore, for CARA bidders, the equilibrium strategy in a FPA is the bid the risk premium associated with the “price gamble” in a SPA!
- Moreover, since $G(x)u\big(x - \gamma(x)\big) = G(x)E[u(x - Y_1)|Y_1 < x]$, the **equilibrium expected utility** (not the bidding strategy!) of a CARA bidder is the same in a FPA and in a SPA.

- In standard auction models with risk-neutral bidders, payoffs are quasi-linear (linear in the payment bidders make) and bidders maximize expected profit with are the difference between expected value and expected payment. **This separation of expected value and expected payment is crucial to the revenue equivalence principle**.

- Risk-averse bidders maximize Expected utility of (Value –Payment). Since utility is concave, the problem is not anymore linear in expected payments.

Qualifications and Extensions

- **4.2 Budget constraints**
  - We assume now that bidders face some form of financial constraints: each bidders has **an absolute budget $\boldsymbol{W_i}$.** If a bidder bids more than w_i and default, a penalty would be imposed.
  - We keep the symmetric independent private value setting. Bidder i value is denoted X_i. Bidders are risk neutral.
  - Each bidder's value-budget pair (X_i, W_i) is IID on $[0,1] \times [0,1]$ according a density f. The pair (x_i, w_i) is the bidder type.
  - A key particularity of this setup is that bidder types are two-dimensional.
  - A bidder's strategy in an auction A is a function $B^A: [0,1] \times [0,1] \rightarrow R$.

Qualifications and Extensions

- **4.2.1 Second-Price Auctions**
- **Proposition 4.2.** *In a second-price auction, it is a dominant strategy to bid according* $B^{II}(x, w) = \min(x, w)$.

- *Proof.*
  - It is dominated to bid above his own budget. If the bidder wins by bidding above his own budget:
    - If the second-highest bid is below w_i, then the bidder wins also by bidding w_i.
    - If the second-highest bid is above w_i, then the bidder wins but makes a loss (the penalty).
  - If $x_i < w_i$, the constraint is not binding. Then, as in the case of absence of budget constraint, it is a weakly dominant strategy to bid x_i.
  - It $x_i > w_i$, a parallel argument shows that bidding w_i dominates bidding less. cqfd

Qualifications and Extensions

- Define now $x" = \min(x, w)$ and consider the type $(x", 1)$:
  - Because values never exceed 1, type $(x", 1)$ is never constraint.
  - Because $\min(x{,}1){=}\mathrm{x} = \min(x, w)$, $B^{II}(x, w) = B^{II}(x", 1)$: thus, in the SPA, type $(x", 1)$ and type (x, w) submits the same bid. In a sense, type $(x", 1)$ is the richest member of the family with types (x, w) such that $\min(x, w) = x$“.
  - Denote $m^{II}(x, w)$ the expected payment of a bidder of type (x, w) in the SPA. Because $B^{II}(x, w) = B^{II}(x", 1)$, we have:

$$m^{II}(x, w) = m^{II}(x", 1) \tag{4.8}$$

  - Define:

$$L^{II}(x") = \{(X, W): B^{II}(X, W) < B^{II}(x", 1)\} \tag{4.9}$$

the set of types who bids less then $(x", 1)$ in a SPA.

Qualifications and Extensions

- Define:

$$F^{II}(x") = \int_{L^{II}(x")} f(X, W)dXdW \qquad (4.10)$$

the probability that a type will outbid one other bidder. The probability that bidder $(x", 1)$ will win the auction is then simply $(F^{II}(x"))^{N-1} \equiv G^{II}(x")$.

- The expected utility of a type $(x", 1)$ when bidding $B^{II}(z, 1)$ is:

$$G^{II}(z)x" - m^{II}(z, 1)$$

- In equilibrium, it is optimal to bid $B^{II}(x", 1)$. By Proposition 3.1, we have:

$$m^{II}(x", 1) = \int_0^{x"} y g^{II}(y)dy \qquad (4.11)$$

Qualifications and Extensions

- By analogy to 2.7, the *ex ante* expected payment of a bidder is a SPA with financial constraint is:

$$E[R^{II}] = \int_0^1 m(x", 1) f^{II}(x", 1) dx" = E\left[{Y_2}^{II(N)}\right] \quad (4.12)$$

Where ${Y_2}^{II(N)}$ is the second-highest of N draws in the distribution F^{II}.

- **4.2.2 First-Price Auctions**

  - Suppose that in a FPA, the equilibrium bidding strategy is:

$$B^I(x, w) = \min(\beta(x), w) \quad (4.13)$$

for some increasing function $\beta(x)$. It must be that $\beta(x) < x$ (otherwise, a bidder with type $x < w$ would bid slightly less. Assume $\beta(x)$ exists.

Qualifications and Extensions

- Define x' to be such that $\beta(x') = \min(\beta(x), w)$ and consider the type $(x', 1)$.
- Bidder $(x', 1)$ never faces a financial constraint. Because $\min(\beta(x'), 1) = \beta(x') = \min(\beta(x), w)$, we have that $B^I(x', 1) = B^I(x, w)$. Thus, in the FPA, the bidder $(x', 1)$= submit the same bid as the bidder (x, w).
- Define:

$$L^I(x') = \{(X, W): B^I(X, W) < B^I(x', 1)\} \quad (4.14)$$

- And m^I, F^I and G^I as in the SPA. We obtain:

$$E[R^I] = E\left[{Y_2}^{I(N)}\right] \quad (4.15)$$

Qualifications and Extensions

- **4.2.3 Revenue Comparison**
  - Note first that since for all $x, \beta(x) < x$, $L^{ii}(x) \subset L^{I}(x)$ (see (4.9) and (4.14)). Equation (4.10) implies moreover that for all $x, F^{I}(x) \leq F^{II}(x)$ (and a strict inequality holds if $x \in [0,1]$. In words, $F^{I}(x)$ stochastically dominates $F^{II}(x)$. This implies that

$$E\left[{Y_2}^{I(N)}\right] > E[{Y_2}^{II(N)}]$$

  - **Proposition 4.3.** *Suppose that bidders are subject to financial constraints. If the first-price auction has a symmetric equilibrium of the form* $B^{I}(x, w) = \min(\beta(x), w)$**,** *then the expected revenue in a first-price auction is greater than the expected revenue in a second-price auction.*

  - Intuition: budget constraint is “softer” in FPA: bidders shade their bid in the FPA without budget constraints. Therefore, budget constraints is less binding in the FPA and more (frequently) binding in the SPA (in which bidders bid their full-value).

Qualifications and Extensions

- **4.3 Asymmetries among bidders**
  - We analyze here a situation in which bidders are *ex-ante* asymmetric: different bidders' values are drawn from **different** distributions.
  - Asymmetries among bidders do not affect bidding behavior in the SPA: it is still a weakly dominant strategy to bid his value.
  - In the FPA, the analysis is far more complicated:
    - The equilibrium exits (see Appendix G) but there isn't a close form solution;
    - The allocation is different: the SPA is efficient and the FPA is not. Therefore, the SPA and the FPA are not revenue equivalent. In fact, no general ranking of revenues can be obtained.

Qualifications and Extensions

- **4.3.1 Asymmetric First-Price Auctions with Two Bidders**
- Setup
  - $N = 2$
  - Values are X_1 and X_2, IID with distribution F_1 on $[0, \omega_1]$ and F_2 on $[0, \omega_2]$
  - Suppose there is an equilibrium of the FPA in which bidders follow β_1 and β_2
  - Suppose moreover that β_1 and β_2 are differentiable, increasing and have inverse $\varphi_1 \equiv \beta^{-1}{}_1$ and $\varphi_2 \equiv \beta^{-1}{}_2$
- Analysis
  - First note that $\beta_1(0) = 0 = \beta_2(0)$: bidding more than his value is dominated.
  - Note also that $\beta_1(\omega_1) = \beta_2(\omega_2)$: indeed, if $\beta_1(\omega_1) > \beta_2(\omega_2)$, bidder 1 would win with probability one when his value is ω_1 and would pay more than he needs to.
  - Conclusion: equilibrium bidding strategies have common range

$$\bar{b} \equiv \beta_1(\omega_1) = \beta_2(\omega_2) \tag{4.16}$$

Qualifications and Extensions

- The expected payoff of bidder i when his value is x_i and he bids less than $\bar{b}$ is:

$$\pi_i(b_i, x_i) = F_j\big(\varphi_j(b)\big)(x_i - b) = H_j(b)(x_i - b)$$

where $H_j(.) = F_j\big(\varphi_j(.)\big)$ is the distribution of bidder j bids.

- The FOC is:

$$h_j(b)(x_i - b) - H_j(b) = 0$$

$$h_j(b)(\varphi_i(b) - b) = H_j(b) \qquad (4.17)$$

where $j \neq i$ and $h_j(b) = H'_j(b) = f_j(\varphi_j(b))\varphi'_j(b)$ is the density of j bids.

Qualifications and Extensions

- This can be rearranged in:

$$\varphi'_j(b) = \frac{F_j(\varphi_j(b))}{f_j(\varphi_j(b))} \frac{1}{(\varphi_i(b)-b)} \quad (4.18)$$

- (4.18) provides a system of differential equation (one for each bidder). With the boundary conditions, this provides an equilibrium of the FPA. An explicit solution can only be derived in a limited number of specific cases.

- WEAKNESS LEADS TO AGGRESSION

  - Suppose that bidder 1 values are stochastically higher than those of bidder 2. We assume that F_1 dominates F_2 in terms of reverse hazard rate:

$$\frac{f_1(x)}{F_1(x)} > \frac{f_2(x)}{F_2(x)} \quad (4.19)$$

Note that reverse hazard rate dominance implies that $F_1(x) < F_2(x)$ (stochastic dominance – see Appendix B). A simple case is $F_1(x) = (F_2(x))^{\theta}, \theta > 1$..

Qualifications and Extensions

- We call bidder 1 the strong bidder and bidder 2 the weak bidder.

- **Proposition 4.4.** *Suppose that the value distribution of bidder 1 dominates that of bidder 2 in terms of reverse hazard rate. Then in a first-price auction, the "weak" bidder 2 bids more aggressively than the "strong" bidder 1. That is, for any* $x \in [0, \omega_2]$, $\beta_1(x) < \beta_2(x)$.

- *Proof.*
  - First, note that if there exists a c such that $0 < c < \bar{b}$ and $\varphi_1(c) = \varphi_2(c) = z$, (4.18) and (4.19) imply that:

$$\varphi'_2(c) = \frac{F_2(z)}{f_2(z)} \frac{1}{(z-b)} > \frac{F_1(z)}{f_1(z)} \frac{1}{(z-b)} = \varphi'_1(c)$$

  - Since $\varphi'_i(c) = \frac{1}{\beta'_i(z)}$, this is equivalent to say that $\beta'_1(z) > \beta'_2(z)$: so, if the curves β_1 and β_2 intersect, they intersect at most once.

Qualifications and Extensions

- By contradiction:
  - Suppose it exists an $x \in [0, \omega_2]$ such that $\beta_1(x) > \beta_2(x)$.
    - Then, either β_1 and β_2 do not intersect and $\beta_1(x) > \beta_2(x)$ everywhere or they intersect only once at some value $z \in [0, \omega_2]$. For all $x \in]z, \omega_2[$, $\beta_1(x) > \beta_2(x)$.
    - In both cases, for x close to ω_2, $\beta_1(x) > \beta_2(x)$.
    - But if $\omega_1 > \omega_2$, from $\beta_1(\omega_1) = \beta_2(\omega_2)$ (see (4.16)). So, $\beta_1(\omega_2) < \beta_2(\omega_2)$. This contradicts the previous claim.
  - Suppose now that $\omega_1 = \omega_2 = \omega$.
    - If $\bar{b} \equiv \beta_1(\omega_1) = \beta_2(\omega_2)$, then for all b close to $\bar{b}$, $\varphi_1(b) < \varphi_2(b)$.
    - Therefore, for all b close to $\bar{b}$, $H_1(b) = F_1(\varphi_1(b)) \leq F_2(\varphi_2(b)) = H_2(b)$.
    - Since $H_1(\bar{b}) = 1 = H_2(\bar{b})$, it must be that $h_1(b) > h_2(b)$.
    - Using (4.17), for all b close to $\bar{b}$, $\varphi_1(b) = \frac{H_2(b)}{h_2(b)} + b > \frac{H_1(b)}{h_1(b)} + b = \varphi_2(b)$.
    - This is a contradiction.

- We know that:
  - F_1 stochastically dominates F_2.
  - For any given value x, bidder 2 bids higher than bidder 1.

Qualifications and Extensions

- What can we say about distributions of bids?
  - Distributions of bids are H_1 and H_2.
  - For all $b \in [0, \bar{b}]$, $\varphi_1(b) > \varphi_2(b)$ (because bidder 2 bids more aggressively).
  - From (4.17) and (4.18):

$$\frac{H_2(b)}{h_2(b)} = \varphi_1(b) - \mathrm{b} > \varphi_2(b) = \frac{H_1(b)}{h_1(b)}$$

  - The distribution of bidder 1 (the strong bidder) dominates the distributions of bidder 2 (the weak bidder) in terms of reverse hazard rate. H_1 dominates therefore H_2 stochastically.

- Why do the weak bidder bids more aggressively than the strong bidder?
  - Two forces are at play:
    - The distribution of values
    - The distribution of bids
  - In equilibrium, the strong bidder has incentives to lower his bid for any given value because, on average, his value dominates the value of the weak bidder.

Qualifications and Extensions

- ASYMMETRIC UNIFORM DISTRIBUTIONS
  - Equilibrium bidding strategies can be derived in asymmetric FPA is the two value distributions are uniform with different supports.
  - We assume:
    - X_1 is uniform with support $[0, \omega_1]$
    - X_2 is uniform with support $[0, \omega_2]$
    - With $\omega_1 \geq \omega_2$
  - Therefore:
    - $F_i(x) = \frac{x}{\omega_i}$
    - $f_i(x) = \frac{1}{\omega_i}$
    - The FOC (4.17) becomes (for $j \neq i$ and $b \in (0, \bar{b})$):

$$\varphi'_i(b) = \frac{\varphi_i(b)}{\varphi_j(b) - b} \quad (4.20)$$

    - This is equivalent to:

$$\left(\varphi'_i(b) - 1\right)\left(\varphi_j(b) - \mathrm{b}\right) = \varphi_i(b) - \varphi_j(b) + \mathrm{b}$$

Qualifications and Extensions

- Adding equations for $i = 1{,}2$, we obtain:

$$\left(\varphi'_1(b) - 1\right)(\varphi_2(b) - b) = \varphi_1(b) - \varphi_2(b) + b$$
$$\left(\varphi'_2(b) - 1\right)(\varphi_1(b) - b) = \varphi_2(b) - \varphi_1(b) + b$$
$$\left(\varphi'_1(b) - 1\right)(\varphi_2(b) - b) + \left(\varphi'_2(b) - 1\right)(\varphi_1(b) - b) = 2b$$
$$\frac{d}{db}\left((\varphi_1(b) - b)(\varphi_2(b) - b)\right) = 2b$$

- Integrating, we obtain:

$$(\varphi_1(b) - b)(\varphi_2(b) - b) = b^2 \qquad (4.21)$$

Note that the constant of integration is equal to zero because $\varphi_i(0) = 0$. And since $\varphi_i(\bar{b}) = \omega_i$ (common range):

$$\left(\omega_1 - \bar{b}\right)\left(\omega_2 - \bar{b}\right) = b^2$$

- So that:

$$\bar{b} = \frac{\omega_1 \omega_2}{\omega_1 + \omega_2} \qquad (4.22)$$

Qualifications and Extensions

- Using (4.21), we note that:

$$\left(\varphi_j(b) - b\right) = \frac{b^2}{\varphi_i(b) - b}$$

- (4.20) can then be rewritten as:

$$\varphi'_i(b) = \frac{\varphi_i(b)(\varphi_i(b) - b)}{b^2} \tag{4.23}$$

- (4.21) is separable in the variables! The next step is to undertake a change of variables by defining $\xi_i(b)$ implicitely by:

$$\varphi_i(b) - b = \xi_i(b)b \tag{4.24}$$

- So:

$$\varphi'_i(b) - 1 = \xi'_i(b)b + \xi_i(b)$$

Qualifications and Extensions

- Using these last results

$$\varphi'_i(b) = \xi'_i(b)b + \xi_i(b) + 1$$

$$\frac{\varphi_i(b)(\varphi_i(b)-b)}{b^2} = \frac{(\xi_i(b)b+b)\ \xi_i(b)b}{b^2} = \frac{\xi_i(b)^2\ b^2+\xi_i(b)\ b^2}{b^2} = \xi_i(b)\ (\xi_i(b)+1)$$

- We obtain therefore:

$$\xi'_i(b)b + \xi_i(b) + 1 = \xi_i(b)\ (\xi_i(b)+1)$$

- Or:

$$\frac{\xi'_i(b)}{\xi_i(b)^2-1} = \frac{1}{b}$$

- The solution is (see notes – error in book!):

$$\xi_i(b) = \frac{1 + k_i b^2}{1 - k_i b^2}$$

where k_i is an integration constant.

Qualifications and Extensions

- Using (4.24), we get (see notes – error in book!):

$$\varphi_i(b) = \frac{2b}{1-k_i b^2} \quad (4.25)$$

and since $\varphi_i(\bar{b}) = \omega_i$, we obtain (see notes – error in book!):

$$k_i = \frac{1}{\omega_j^2} + \frac{1}{\omega_i \omega_j} \quad (4.26)$$

- The bidding strategies is obtained by investing (4.25) (see notes – error in book!):

$$\beta_i(x) = \frac{1}{k_i x}\left(1 + \sqrt{1 + k_i x^2}\right) \quad (4.27)$$

To verify that it is an equilibrium, you must check that $\beta_i(x)$maximizes expected profits. This does NOT prove that it is the only equilibrium however.

Qualifications and Extensions

- **4.3.2 Revenue Comparison**
  - We first show that FPA may exceed that from a SPA.

  - **Example 4.3.** *With asymmetric bidders, the expected revenue in a FPA auction may exceed that in a SPA.*
    - Assume:
      - $\alpha \in [0,1)$
      - Bidder 1 value X_1 is uniform on $\left[0, {}^{1}/_{(1-\alpha)}\right]$
      - Bidder 2 value X_2 is uniform on $[0, {}^{1}/_{(1+\alpha)}]$
    - Note that when $\alpha = 0$, the situation is symmetric and the FPA and SPA yield the same revenue.

  - REVENUE IN THE SPA
    - It is a dominant strategy to bid one's value.
    - The distribution of the selling price is therefore:

$$L_\alpha{}^{II}(p) = Prob[\min(X_1, X_2) \leq p]$$

with $p \in [0, {}^{1}/_{1+\alpha}]$

Qualifications and Extensions

- This is equal to:

$$\begin{aligned} Prob[\min(X_1, X_2) \leq p] \\ &= \Pr(X_1 \leq \text{p and } X_2 > p) + \Pr(X_1 > \text{p and } X_2 \leq p) + \Pr(X_1 \leq \text{p and } X_2 \leq p) \end{aligned}$$

$$= F_1(p)\big(1 - F_2(p)\big) + F_2(p)\big(1 - F_1(p)\big) + F_1(p)F_2(p)$$

Therefore:

$$\begin{gathered} L_\alpha{}^{II}(p) = F_1(p) + F_2(p) - F_1(p)F_2(p) \\ = (1-\alpha)p + (1+\alpha)p - (1-\alpha)(1+\alpha)p^2 \\ = 2p - (1-\alpha^2)p^2 \end{gathered}$$

over $[0,2]$.

- $L_\alpha{}^{II}(p)$ is increasing in α: the expected selling price in the SPA when $\alpha > 0$ is **lower** than the expected selling price when $\alpha = 0$.

Qualification and Extensions

- REVENUE IN THE FPA (see note – error in book!)
  - Since $\omega_1 = {}^{1}/_{(1-\alpha)}$ and $\omega_2 = {}^{1}/_{(1+\alpha)}$, from (4.22) $\bar{b} = \frac{\omega_1 \omega_2}{\omega_1 + \omega_2}$, it follows that $\bar{b} = {}^{1}/_{2}$.
  - Constants of integration in (4.26) are $k_1 = 2(1+\alpha)$ and $k_2 = 2(1-\alpha)$.
  - Using (4.25), the inverse bidding strategies are:

$$\varphi_1(b) = \frac{2b}{1 - 2(1+\alpha)b^2}$$
$$\varphi_2(b) = \frac{2b}{1 - 2(1-\alpha)b^2}$$

  - The distribution of the equilibrium prices in the FPA is therefore:

$$L^{I}{}_{\alpha}(p) = Prob[\max(\beta_1(X_1), \beta_2(X_2)) \leq p]$$

where $p \in [0, \frac{1}{2}]$.

Qualifications and Extensions

- We have:

$$L^I{}_\alpha(p) = F_1(\varphi_1(p)) \times F_2(\varphi_2(p)) = \frac{(1-\alpha^2)(2p)^2}{1-\alpha^2(2p)^4}$$

→ $L^I{}_\alpha(p)$ is decreasing in α: the expected selling price when $\alpha > 0$ is **higher** than the expected selling price when $\alpha = 0$.

- We have shown than in for all $\alpha > 0$, in this example, the expected selling price in the FPA is greater than that in the SPA. The result holds generally with uniformly distributed values.

- **Example 4.4.** *With asymmetric bidders, the expected revenue in a SPA may exceed that in a FPA.*
  - Suppose bidder 1 value X_1 is distributed according to $F_1(x) = x - 1$ over $[1,2]$ and bidder 2 value X_2 is distributed according to $F_2(x) = \exp\left(\frac{1}{2}x - 1\right)$ over $[0,2]$.

Qualifications and Extensions

- Note that:
  - The lowest value value for bidder 2 is not anymore 0.
  - F_2 has a mass point at 0 (because $F_2(0) \neq 0$).

- Then:
  - In the FPA:
    - Equilibrium bidding strategies in the range [0,1] for fhe FPA are:
      - $\beta_1(x) = x - 1$
      - $\beta_2(x) = \frac{1}{2}x$
    - The distribution of prices is: $L^I(p) = p \ \exp(p-1)$.
    - The expected revenue is: $E[R^I] = 0.632$.
  - In the SPA:
    - The distribution of prices over $[0,2]$ is: $L^{II}(p) = \begin{cases} \exp\left(\frac{1}{2}p - 1\right) & if \ \ p \leq 1 \\ (p-1) + (2-p)\exp\left(\frac{1}{2} - 1\right) & if \ p > 1 \end{cases}$
    - The expected revenue is: $E[R^{II}] = 0.662$.

Qualifications and Extensions

- **4.3.2 Efficiency Comparison**
  - In the SPA, it is a weakly dominant strategy to bid his value:
    - The winner is the one with the highest value
    - The SPA is always ex-post efficient (under the assumption of private values)
  - In contrast, asymmetries lead to inefficient allocations in a FPA
    - Suppose there are 2 bidders and (β_1, β_2) is an equilibrium such that both strategies are continuous and increasing;
    - Suppose that $\beta_1(x) < \beta_2(x)$;
    - Because both strategies are continuous, for some small ε, it will also be the case that $\beta_1(x + \varepsilon) < \beta_2(x - \varepsilon)$;
    - The allocation is therefore inefficient with positive probability since bidder 2 would win the auction even though he has a lower value.

  - **Proposition 4.5.** *With asymmetrically distributed private values, a SPA always allocates the object efficiently, whereas with positive probability, a FPA does not.*

Qualifications and Extensions

- **4.4 Resale and Efficiency**
  - Achieving an efficient allocation may be a primary policy goal of the seller (especially for public authorities).
  - During a long period, it has be argued that postauction transactions among buyers (resale) will result in an efficient final allocation. The conclusion is that the choice of the auction is then irrelevant (from the efficiency point of view). The seller may then focus only on revenue maximization.

  - To examine the resale question, consider the following basic setup:
    - 2 bidders with value X_1 and X_2
    - Values are independently distributed according to F_1 and F_2, with common support $[0, \omega]$ and with $F_1 \neq F_2$ (source of asymmetry)
    - Suppose also that $E[X_1] \neq E[X_2]$

Qualifications and Exensions

- Suppose that β_1 and β_2 are equilibrium bidding strategies in the FPA and that we know they are increasing.
- Suppose also that the conclusion of the auction (both bids) are publicly announced. Values are then common knowledge at that time because $x_1 = \beta^{-1}{}_1(b_1)$ and $x_2 = \beta^{-1}{}_2(b_2)$.

- If $b_1 > b_2$ but $x_1 < x_2$, the auction is inefficient. But because values are common knowledge, there are some unrealized gains from trade. In particular, bidder 1 could offer to resell the object to bidder 2 to some price between x_1 and x_2.

- Despite its apparent intuitiveness, this line of reasoning fails to take into account that rational buyers will behave differently during the auction once they know that resale is possible.

Qualifications and Extensions

- To model resale, we must be more specific about how resale takes place:
  - Suppose that the auction winner, after learning the loosing bid, may resell the object to the other bidder by making a one-time take-it-or-leave-it offer.
  - In this setup, the bargaining power is in the hands of the auction winner (the new owner of the object).
  - If $x_1 < x_2$, bidder 1 will rationally offer the object to bidder 2 at a price $p = x_2$, or just bellow. Bidder 1 will make profit $x_2 - b_1$ and bidder 2 profit will be 0 (or just above).
  - This setup makes the best case for resale since:
    - It ensures efficiency if values are commonly known;
    - It assumes no transaction costs or delays.
  - **The main result is that it can't be an equilibrium of the FPA with resale in which the auction outcome completely reveals the values.**

Qualifications and Extensions

- Proof.
  - Suppose that the FPA with resale has an efficient equilibrium.
  - The equilibrium would be:
    - β_1 and β_2, the bidding strategies in the FPA, increasing, with inverse φ_1 and φ_2
    - At the resale stage, if announced bids are b_1 and b_2 are such that $b_i > b_j$ but $x_i < \varphi_j(b_j)$, then i makes a take-it-or-leave-it offer to sell the object at j at price $\varphi_j(b_j)$. The offer is accepted if $x_j \geq \varphi_j(b_j)$.
  - Assuming that β_1 and β_2 are invertible means that the FPA, the values will commonly known. Therefore, if there are unrealized gains from the trade, the resale will take place and the object will be allocated efficiently.
  - Note that, as previously, in the FPA, β_1 and β_2 must have common range:
    - $\beta_1(0) = 0 = \beta_2(0)$
    - For some $\bar{b}$, $\beta_1(\omega) = \bar{b} = \beta_2(\omega)$
  - Suppose that:
    - Bidder 2 behaves according to the equilibrium
    - Bidder 1 deviates and bids as if he has value z_1 (instead of x_1). Bidder 1 overall expected payment will be in equilibrium:

$$m_1(z_1) = F_2(\varphi_2(\beta_1(z_1)))\beta_1(z_1) - \int_{z_1}^{\varphi_2(\beta_1(z_1))} \max(z_1, x_2)\, f_2(x_2) dx_2 \qquad (4.29)$$

Qualifications and Extensions

- $\int_{z_1}^{\varphi_2(\beta_1(z_1))} \max(z_1, x_2)\, f_2(x_2) dx_2$ is the expected resale product:
  - If $z_1 < X_2 < \varphi_2(\beta_1(z_1))$: bidder 1 wins the auction and resells the object to bidder 2 at a price of $X_2 = \max(z_1, X_2)$
  - If$\varphi_2(\beta_1(z_1)) < X_2 < z_1$: bidder 1 loses the auction and purchases the object from bidder 2 at a price of $z_1 = \max(z_1, X_2)$
- So, if bidder 1 behaves as his value is z_1, the probability that he will get the object is $F_2(z_1)$ (because the final allocation is efficient)
- The bidder 1 expected payoff is then:

$$F_2(z_1)x_1 - m_1(z_1)$$

- If equilibrium, it must not be optimal to deviate. So:

$$F_2(x_1)x_1 - m_1(x_1) \geq F_2(z_1)x_1 - m_1(z_1)$$

- The F.O.C. for this optimization is:

$$m_1(x_1) = x_1 f_2(x_1)$$

Qualifications and Extensions

- Since $m_1(0) = 0$:

$$m_1(x_1) = \int_0^{x_1} x_2 f_2(x_2) dx_2 \tag{4.30}$$

- Setting $z_1 = x_1$ in (4.29) and equation with (4.30), we obtain a necessary condition for the FPA with resale to be efficient: for all x_1:

$$F_2(\varphi_2(\beta_1(x_1)))\beta_1(x_1) - \int_{z_1}^{\varphi_2(\beta_1(x_1))} \max(x_1, x_2) f_2(x_2) dx_2 = \int_0^{x_1} x_2 f_2(x_2) dx_2 \tag{4.31}$$

- (4.31) says that the expected payment of bidder 1 in the equilibrium of the FPA with resale is the same as the in the SPA. This is an extension of the revenue equivalence principle to the asymmetric case:
  - Same outcomes (under efficiency)
  - Same expected payment

Qualifications and Extensions

- Now, note that:
  - $\beta_1(\omega) = \bar{b}$, so $\varphi_2(\beta_1(\omega)) = \omega$
  - Setting $x_1 = \omega$ in (4.31) leads to $\bar{b} = E[X_2]$
  - Interchanging bidder 1 and 2, we obtain $\bar{b} = E[X_1]$
- Since $E[X_1] \neq E[X_2]$, this is a contradiction.
- We have shown therefore that in a asymmetric FPA with resale, bidding strategies cannot be increasing everywhere. This means that equilibrium behavior in the FPA cannot reveal values completely. Resale transactions must take place under incomplete information.
- Reaching an efficient allocation in all circumstances under this incomplete information is impossible:
  - Because equilibrium bidding strategies in the FPA are not increasing everywhere, bids don't reveal completely values;
  - So, at a given bid of bidder i can correspond a range of values;
  - The best (or rational) estimation that bidder j can do of bidder i value in this case the expected value of bidder i value on this range;
  - Bidder j will therefore make a take-it-or-leave it offer when his value is bellow the expected value of bidder i on the range;
  - But the real value of bidder i may be in fact below or above the one of bidder j.